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LAW OF ERROR IN THE POSITION OF A POINT IN SPACE.

BY E. L. DE FOREST.

[Continued from page 40.]

WHEN the positions of the coefficients in the given polynomial and its expansion are referred to any arbitrarily chosen rectangular axes, the differential equations (15) will still be those of the limiting function sought, provided that the constants α, β, &c., are understood to refer to the axes thus adopted, each L receiving new sub-indices equal to its new coordinates, expressed in the unit of measure $\Delta x = \Delta y = \Delta z$. For convenience, we shall henceforth assume the free axes of the system of weights L as the coordinate axes, because this, as already stated, reduces α_1, α_2, α_3 and γ_1, γ_2, γ_3 to zero, so that the equations (15) take the simple form

$$\beta_1 d_x w = \left(\frac{-i}{n+1}\right) w, \quad \beta_2 d_y w = \left(\frac{-j}{n+1}\right) w, \quad \beta_3 d_z w = \left(\frac{-k}{n+1}\right) w. \qquad (19)$$

Substituting for i, j, k their equivalents from (13), and using n instead of $n+1$, which we may do because n is infinite, we get

$$\frac{d_x w}{w} = \frac{-x dx}{n\beta_1 (dx)^2}, \qquad \frac{d_y w}{w} = \frac{-y dy}{n\beta_2 (dy)^2}, \qquad \frac{d_z w}{w} = \frac{-z\, dz}{n\beta_3 (dz)^2}. \qquad (20)$$

If we also write

$$r_1^2 = \beta_1 (dx)^2, \qquad r_2^2 = \beta_2 (dy)^2, \qquad r_3^2 = \beta_3 (dz)^2, \qquad (21)$$

$$h_1^2 = 1 \div 2nr_1^2, \qquad h_2^2 = 1 \div 2nr_2^2, \qquad h_3^2 = 1 \div 2nr_3^2, \qquad (22)$$

then (20) will stand

$$\frac{d_x w}{w} = -2h_1^2 x dx, \quad \frac{d_y w}{w} = -2h_2^2 y dy, \quad \frac{d_z w}{w} = -2h_3^2 z dz. \qquad (23)$$

From the significations of β_1, β_2, β_3, as stated in connection with (15), and the fact that $\Sigma L = 1$, it appears that r_1, r_2, r_3 in (21) represent the "radii of gyration" of the system of coefficients or masses L with respect to the free-axial planes of YZ, XZ and XY, in the sense in which that nomenclature was used by me in ANALYST, May, '80, p. 80.

The squared radius of gyration of a system of masses of material points, with respect to a plane, was there defined as the result obtained by multiplying the mass of each point into the square of its perpendicular distance from the plane, adding all the products together, and dividing their sum by the sum of the masses. The squared radius of such a system with respect to a line or point is found in like manner, using the distance of each mass from the line or point. When the points and their masses represent points of error and their probabilities, the radius of gyration of the system of masses with respect to a plane, a line, or a point, is identical with the *quadratic mean error* or q. m. deviation of the syst. of points, from such plane, line or point. It will be seen that in forming the mean of the squares of the errors, each error is taken with a weight proportional to the probability of its occurrence. It was shown in my article that when two entire polynomials and their product are arranged each in block form, and coordinate planes are passed through the centre of gravity of the coefficients in each and parallel to the sides of the block, the square of the radius of gyration of the system of coefficients in the product, with respect to any one of these coordinate planes, is equal to the sum of the squares of the radii for the two factors, with respect to their corresponding planes. It is not material, however, whether the rectangular coordinate planes are parallel to the sides of the block, or whether they have other directions, for, as we have seen, the system of coefficients in a block-formed entire polynomial can be replaced, to any desired degree of approximation, by those of another block whose centre and sides have any required position and directions, the intervals being supposed equal and as small as we please, while the exponents are changed so that in any term they shall represent the number of new intervals from the coefficient of that term to the coordinate planes respectively.

A relation between distances, such as lever arms or radii of gyration, holds true without regard to the magnitude of the common unit employed to measure such distances. If any two polynomials have had their coordinate planes of reference changed as above, the coefficients in their product are not changed either in magnitude, or in their relative positions when set so that the coordinates of each are represented by the corresponding exponents.

If the axes in each factor are taken through the centre of gravity of the coefficients, the origin or place of exponent zero in the product will be the centre of gravity of its system of coefficients. The proposition then holds good, that the square of the radius of gyration of the system of coefficients in the product, with respect to any one of the coordinate planes, is equal to the sum of the squares of the corresponding radii in the two factors. If such a polynomial is raised to the *n*th power, the square of the radius of gyration of the coefficients in the expansion, with respect to any one of the coordinate planes, is n times the square of the corresponding radius in the first power. This shows that in (22), nr_1^2, nr_2^2 and nr_3^2 represent the squared radii of gyration of the system of coefficients l in the expansion, with respect to the planes of YZ, XZ and XY. In other words,

$$\varepsilon_1 = r_1\sqrt{n}, \qquad \varepsilon_2 = r_2\sqrt{n}, \qquad \varepsilon_3 = r_3\sqrt{n}, \tag{24}$$

are the quadratic mean errors or q. m. deviations of the coefficients in the expansion, from the coordinate planes which pass through the free axes.

When two polynomials referred to rectangular planes through their centres of gravity are multiplied together, their product being likewise referred, the sum of the squares of the two radii of gyration of the system of coefficients in the product, with respect to the ZX and the ZY planes, is equal to the sum of the squares of the radii in the two factors, with respect to the same two planes in each. But the square of the distance of any point from the axis of Z is equal to the sum of the squares of its distances from the ZX and ZY planes, so that the square of the radius of gyration of any system of coeff's, about the Z axis, is equal to the sum of the squares of its radii with respect to the ZX and ZY planes. Hence the square of the radius of gyration of the coeff's in the product of the two polynomials, about the Z axis of the product, is equal to the sum of the squares of the radii for the two factors, about their Z axes. The like will evidently be true for radii about the X or Y axes. If a polynomial thus referred is raised to the nth power, the square of the radius of gyration for the product, about any one of the coordinate axes, is n times the square of the radius for the first power, about the same axis. It is a known mechanical property of the free axes, in bodies of three dimensions, that out of all axes of rotation which can be passed through the centre of gravity, the one which renders the radius of gyration a minimum is one of the free axes, and the one which renders it a maxim'm is another free axis. If therefore a given polynomial is raised to successive powers, making $n = 2$, $n = 3$, &c., and the polynomial and its powers are all referred to the same rectangular axes arbitrarily taken through the centre of gravity of the coefficients in the first power, then in the nth power the centre of gravity remains at the origin, and the radius of gyration about each axis is $\sqrt{n}$ times what it was in the first power, for any given directions of the axes, so that it is still a minimum when taken about one of the free axes of the first power, and a maximum about another. Hence, the free axes retain a constant position in all the successive powers. The free axes of the nth system of coefficients are the same as those of the first, and the squares of the radii of gyration about them are

$$n(r_2^2+r_3^2), \qquad n(r_1^2+r_3^2), \qquad n(r_1^2+r_2^2). \tag{25}$$

The squared radius of gyration of the coefficients in the nth power, with respect to the origin or centre of gravity, is

$$E^2 = n(r_1^2 + r_2^2 + r_3^2). \tag{26}$$

We now proceed to integrate the eq's (23), and find that the func. sought is

$$w = ce^{-(h_1^2x^2+h_2^2y^2+h_3^2z^2)}, \tag{27}$$

from which (23) can be derived by differentiating with respect to the variables separately, and dividing by w. To find the value of c we consider that since the sum of the coefficients in any power of the polynomial is unity, we shall have at the limit,

$$\frac{1}{dx\,dy\,dz}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w\,dx\,dy\,dz = 1, \tag{28}$$

which is equivalent to

$$\frac{c}{dx\,dy\,dz}\int_{-\infty}^{+\infty} e^{-h_1^2x^2}\,dx\int_{-\infty}^{+\infty} e^{-h_2^2y^2}\,dy\int_{-\infty}^{+\infty} e^{-h_3^2z^2}\,dz = 1.$$

The known values of the three definite integrals here are

$$\sqrt{\pi}\div h_1, \qquad \sqrt{\pi}\div h_2, \qquad \sqrt{\pi}\div h_3,$$

so that we have

$$\frac{c}{dx\,dy\,dz}\left(\frac{\pi^{\frac{3}{2}}}{h_1h_2h_3}\right) = 1, \tag{29}$$

which determines c, and the complete equation of the limiting funct. stands

$$w = \frac{h_1h_2h_3\,dxdydz}{\pi^{\frac{3}{2}}}\,e^{-(h_1^2x^2+h_2^2y^2+h_3^2z^2)}. \tag{30}$$

If we assign any constant value to w, the exponent of e becomes constant, so that

$$h_1^2x^2+h_2^2y^2+h_3^2z^2 = C. \tag{31}$$

This is the equation of an ellipsoid whose centre and axes coincide with the origin and coordinate axes. The axes of the ellipsoid lying in the x, y, z directions are inversely proport'al to h_1, h_2, h_3, and therefore by (22) are directly propor'l to r_1, r_2, r_3. Since any change in w, and consequently in C, causes all three axes to vary in like proportion, it appears that the loci of points of error whose probabilities w are the same, are ellipsoidal surfaces which are all similar and similarly situated and concentric. From their symmetry with respect to the coordinate axes, it follows that these are the free axes of the system of expanded coefficients or probabilities w, regarded as masses in space, so that the given probabilities L, and the resultant probabilities w, have both the same free axes. This accords with what has been already found in connection with (25). The value of w is a maximum at the origin, which is the centre of gravity of the whole system of masses w, as it also is that of the masses L. The function (30) represents the law of facility of deviation of the centre of gravity of a large number n of similarly observed points of error, from its most probable place. For w is the probability that the sums of the deviations in the x, y, z directions will be $x = idx$, $y = jdy$, $z = kdz$ respectively, and this is the probability that their arithmetical means will be $x \div n$, $y \div n$, $z \div n$ respectively.

If all the possible points of error are set closer together, so that the intervals dx, dy, dz are reduced to $dx \div n$, $dy \div n$, $dz \div n$, the distribution of the probabilities or masses w will represent the law of facility of deviation of the mean position or centre of gravity of the n observed points, from its most probable place. Thus, whatever may be the actual law for a single observed point, as expressed by the distribution of the probabilities L in (3), the law of facility of error of the centre of gravity of a large number n of similarly observed points is always expressed by the same exponential form of function as in (30). This function, then, seems to be a typical form of the law of probability of errors in space, and may reasonably be assumed as the most plausible law for all such observations, in the absence of any previous knowledge to the contrary. Although the law has hardly any direct practical applications, it is important in theory. Its properties are quite analogous to those of the law of errors in two dimensions, as discussed by me in ANALYST, May, '81. Indeed the law for three dimensions is the general law of probability, which includes the laws for one and two dimensions as special cases under it

If an unlimited number of equidistant planes are supposed to be drawn parallel to the coordinate planes, dividing space into elementary cubes $dxdy$ dz, so situated that the origin is at the middle of one of them, then (30) may be regarded as giving the probability w that a point of error which occurs will fall within the cube whose centre is at the point x, y, z.

The like holds true approximately when any arbitrary but small finite values are assigned to dx, dy and dz, for the probability that an error will fall within any small space whose location is given is approximately proportional to the size of that space. From observed positions of a considerable number of points of error, regarded as material points whose masses are equal, we can find the positions of the free axes of the system of points, and then compute the quadratic mean deviation of the system from each of the free-axial coordinate planes. These are approximately the values of ε_1, ε_2, ε_3 in (24), from which with (22) we find

$$h_1^2 = 1 \div 2\varepsilon_1^2, \qquad h_2^2 = 1 \div 2\varepsilon_2^2, \qquad h_3^2 = 1 \div 2\varepsilon_3^2. \tag{32}$$

It is not material what values are assigned to dx, dy and dz, nor whether they are taken equal to each other, only being represented as infinitesimals in the formula, they ought always to be quite small in comparison with the q. m. deviations ε_1 &c., which are finite. If the probability of deviation from the most probable point is considered to be the same in all directions, we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, and by (24) and (26) the q. m. deviation measured directly from the origin is

$$E = \varepsilon_1\sqrt{3}. \tag{33}$$

Consequently by (32),

$$h_1^2 = h_2^2 = h_3^2 = 1 \div \tfrac{2}{3}E^2, \tag{34}$$

and the function (30) becomes

$$w = dxdydz\left(\frac{h_1}{\sqrt{\pi}}\right)^3 e^{-h_1^2 r^2}, \tag{35}$$

where r is the radius vector or distance from the origin to the centre of the space $dxdydz$, and h_1 is computed by (34), from the observed value of E.

Returning now to the most general form of the function, let (30) be referred to polar coordinates by taking

$$x = r\cos\theta\cos\varphi, \qquad y = r\cos\theta\sin\varphi, \qquad z = r\sin\theta, \tag{36}$$

where θ is the angle which r makes with the XY plane and φ is the angle which its projection on that plane makes with the X axis. Then if we put

$$a^2 = h_1^2\cos^2\theta\cos^2\varphi + h_2^2\cos^2\theta\sin^2\varphi + h_3^2\sin^2\theta, \tag{37}$$

(30) becomes

$$w = \frac{h_1 h_2 h_3\, dxdydz}{\pi^{\frac{3}{2}}} e^{-a^2 r^2}. \tag{38}$$

This expresses the facility of error at any point along an unlimited radius vector whose direction is given. From the origin as a centre, suppose a series of spherical surfaces to be described at infinitely small intervals eq'l to

$$(dx\,dy\,dz)^{1/3}.$$

Let the origin be made the apex of a square pyramid whose axis coincides with r, and whose base, or intersection with the spherical surface of unit radius, is the infinitesimal $(d\psi)^2$, so that if the pyramid is extended to the dist. r its base is $(rd\psi)^2$. The section of the pyramid included between consecutive spherical surfaces at distance r is

$$(rd\psi)^2(dx\,dy\,dz)^{1/3},$$

and $dxdydz$ is contained in this $(rd\psi)^2 \div (dx\,dy\,dz)^{2/3}$ times, so that the probability that a point of error which occurs will fall within this section of the pyramid, is

$$w(rd\psi)^2 \div (dxdydz)^{2/3},$$

and the probability that it will fall anywhere within the pyramid whose height is r will be

$$p = \frac{1}{dr}\int_0^r w\left(\frac{rd\psi}{(dx\,dy\,dz)^{1/3}}\right)^2 dr,$$

where $dr = (dxdydz)^{1/3}$. Taking w from (38) we get

$$p = \frac{h_1 h_2 h_3 (d\psi)^2}{\pi^{\frac{3}{2}}}\int_0^r e^{-a^2 r^2} r^2 dr. \tag{39}$$

The probability P that the error will fall within the pyramidal space extended to infinity, is found by taking ∞ as the upper limit of the integral. If now it is required that the probability p for the finite space shall be a given fraction ν of the prob. P for the infinite space, we have the condition

$$\int_0^r e^{-a^2r^2}r^2dr \div \int_0^\infty e^{-a^2r^2}r^2dr = \nu. \tag{40}$$

The known value of the integral in the denominator is $\sqrt{\pi} \div 4a^3$. As for the numerator, we have

$$e^{-a^2r^2} = 1-\tfrac{1}{1}(ar)^2+\tfrac{1}{1.2}(ar)^4-\tfrac{1}{1.2.3}(ar)^6+\&c.,$$

$$\int_0^r e^{-a^2r^2}r^2dr = r^3\left\{\frac{1}{3}-\frac{1}{5.1}(ar)^2+\frac{1}{7.1.2}(ar)^4-\frac{1}{9.1.2.3}(ar)^6+\&c.\right\}.$$

Hence (40) becomes

$$\frac{4(ar)^3}{\sqrt{\pi}}\left\{\frac{1}{3}-\frac{1}{5.1}(ar)^2+\frac{1}{7.1.2}(ar)^4-\frac{1}{9.1.2.3}(ar)^6+\&c.\right\} = \nu. \tag{41}$$

If any particular value is assigned to ν, we have a numerical equation containing only one unknown quantity ar, the value of which may be found by known methods. If for instance we wish to find a limit such that it is an even chance whether the point of error will fall within or without it, taking $\nu = \frac{1}{2}$, and using 12 terms of the series, (41) gives

$$ar = 1.087652,$$

the last decimal figure being doubtful. This determines the distance r of the desired limit from the origin, in any given direction which determines a. To find the locus of the limit for all directions, since $-(ar)^2$ is the exponent of e in (30), we have

$$h_1^2x^2+h_2^2y^2+h_3^2z^2 = (1.087652)^2,$$

and assigning to h_1, h_2, h_3 their equivalents from (32),

$$\left(\frac{x}{1.53817\,\varepsilon_1}\right)^2+\left(\frac{y}{1.53817\,\varepsilon_2}\right)^2+\left(\frac{z}{1.53817\,\varepsilon_3}\right)^2 = 1. \tag{42}$$

This is the equation of an ellipsoid whose semi-axes coincide with the coordinate axes, and are equal to 1.53817 times the q. m. deviations from the YZ, XZ and XY planes. It is the *ellipsoid of probable error*, since there is an even chance that any error which occurs will fall within it.

If any three or more planes are made to meet at the origin, so as to include between them any polyhedral angle, there is an even chance whether a point of error which falls within this angle will fall inside or outside of the ellipsoid. If the probability of a given deviation or error is the same in all directions, we may introduce the direct central q. m. error E from (33), and (42) becomes the equation of a sphere,

$$x^2 + y^2 + z^2 = (.88806\,E)^2, \tag{43}$$

whose radius .88806 E is the probable error, or probable deviation of the observed point from its most probable place.

We can now make a comparison between the relations which the probable error bears to the central q. m. error, in space of one, two, or three dimensions. In the ordinary case of one dimension, it is well known that

prob. error = .67449 (q. m. error).

From the equation of the "ellipse of probable error" in the case of two dimensions (ANALYST, May '81, p. 77), and the fact that when errors are equally probable in any direction the q. m. deviation from the origin is $\sqrt{2}$ times the q. m. deviation from either the X or the Y axis, it follows that

prob. error = 1.17741 (q. m. e.)$\div\sqrt{2}$ = .83255 (q. m. e.).

Taking these two cases in connection with (43), we see that when the probability of an error of given amount is the same in all the possible directions in any one of the three cases, the ratio of the probable error to the q. m. error will be

.67449, 83255, .88806,

according as the errors are limited to space of one, two, or three dimensions.

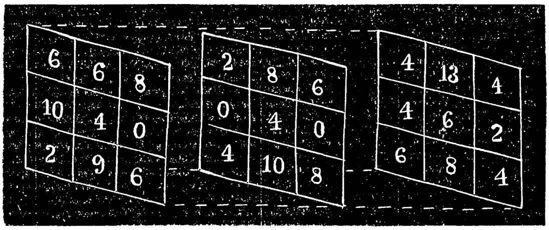
The occurrence of errors in three dimensions can be illustrated by the drawing of balls from an urn, each ball being marked with three numbers representing the x, y and z coordinates of a possible point of error, referred to the true point, or place of zero error, as an origin. Each ball is restored to the urn as soon as drawn, and they are all mixed so as to keep the chances the same. Knowing the number of balls of each kind, we have the prob'ty of drawing one of that kind, and all these probabilities are coefficients L in a polynomial such as (3), the numbers marked on the balls becoming the exponents of ξ, η, ζ. When n drawings are made, the probability that the sums of the x . y-, and z- numbers drawn, will be s, t and v respectively, is the coefficient of $\xi^s\eta^t\zeta^v$ in the expansion of the polynomial to the nth power.

If n is a large number, the probability of any given result can be found by means of the limiting function (30). To obtain this, we should proceed in a manner analogous to that given for the case of errors in space of two dimensions, at p. 80 of my ANALYST article cited. It is easy to find the centre of gravity of the coefficients L, and their coordinates referred to axes passing through this centre and parallel to the original axes. Then known formulas such as are given in Poisson's treatise already cited enable us to calculate the positions of the free axes, or principal axes through the centre of gravity, and to find the radii of gyration r_1, r_2, r_3 of the system of coefficients L about these axes, from which h_1, h_2, h_3 are obtained by (22), and the limiting function (30) is determined. This function has its maximum value at the origin, which is the centre of gravity of the coefficients in the expansion, and the relative position of this centre is readily found because its coordinates, referred to the original axes, are n times those of the centre of gravity of the coefficients L in the first power. By assigning the proper values to x, y, z in (30), the approximate value w of the coefficient of any term in the expansion may be found. It will be understood that $dx = dy = dz$ here represents an interval equal to the distance, in any original coordinate direction, between coefficients of terms whose corresponding exponents differ by unity. For convenience, we may reckon

$$dx = dy = dz = 1,$$

making this the unit of measure for x, y, z and r_1, r_2, r_3.

For the purpose of illustrating and verifying results contained in this paper, a polynomial of 27 terms has been constructed, whose coefficients arranged in block form are shown in the accompanying diagram. The sum of all the numbers in it is 144, so that the true coefficients L are the numbers in the diagram divided by 144, and the sum of them all is unity. For greater simplicity, the numbers have been so chosen that when regarded as the masses of material points, their centre of gravity and free axes coincide with the centre and axes of symmetry of the block. By the notation we have used, the three coefficients in the left hand column, for example, are



$$L_{-1,1,1} = \tfrac{6}{144}, \qquad L_{-1,1,0} = \tfrac{10}{144}, \qquad L_{-1,1,-1} = \tfrac{2}{144}.$$

These in the lower right hand row are

$$L_{-1,-1,-1} = \tfrac{6}{144}, \qquad L_{0,-1,-1} = \tfrac{8}{144}, \qquad L_{1,-1,-1} = \tfrac{4}{144}.$$

The whole polynomial is denoted by

$$\Sigma_{c=-1}^{c=1}\, \Sigma_{b=-1}^{b=1}\, \Sigma_{a=-1}^{a=1}\, (L_{a,b,c}\xi^a\eta^b\zeta^c).$$

When the polynomial is raised to the second power, and referred to the same axes as in the first power, we find that the coefficients, forming a block of 125 terms, have their centre of gravity and free axes coincident with the origin and axes of reference, and these are the centre and axes of symmetry of the block, just as they were in the first power. It is also found that the squared radius of gyration of the coefficients with respect to the origin, or with respect to any one of the three axes, or to any one of the three coordinate planes of reference passing through them, is twice as great in the sec'd power as it is in the first.

To obtain approximately the coefficients w in the expansion of the polynomial to any high power, say the nth, we can proceed thus. The coeff's L give

$$\beta_1 = \tfrac{76}{144}, \qquad \beta_2 = \tfrac{102}{144}, \qquad \beta_3 = \tfrac{114}{144},$$

and by (21), (22) and (13),

$$h_1 dx = 6 \div \sqrt{(38n)}, \quad h_2 dy = 6 \div \sqrt{(51n)}, \quad h_3 dz = 6 \div \sqrt{(57n)},$$

$$h_1^2 x^2 = 36i^2 \div 38n, \quad h_2^2 y^2 = 36j^2 \div 51n, \quad h_3^2 z^2 = 36k^2 \div 57n.$$

Substituting these in (30) we find

$$\log w = \bar{1}.06711 - \tfrac{3}{2}\log n - \frac{1}{n}(.41143 i^2 + .30656 j^2 + .27429 k^2),$$

from which the values of w may be computed by assigning suitable values to i, j, k, which in this example are any whole numbers, either + or — It would be interesting to have the polynomial raised algebraically to a power high enough to show the agreement between the true coefficients in the expansion, and their approximate values as given by the above formula; but this is impracticable, owing to the tedious length of the work required in forming the algebraic expansion.

The exponential function (30), regarded as the law of probability of error in space, has been reached by previous writers, but in ways quite different from ours. One of the early investigators of the law for space of two and three dimensions was Bravais, whose essay, *Sur les probabilité's des erreurs de situation d' un point*, may be found in the *Mémoires . . . par divers savans . . . .*, Inst. France, Vol. IX. (1846). His process treats the probability of error of an observed point in one direction as independent of its probability of error in another direction perpendicular to the first. This objectionable assumption, which has been made by other writers on the subject so far as I know, is avoided in our present method. (See ANALYST, May 1881, pp. 75 and 79.)

ON THE COMPUTATION OF PROBABLE ERROR.

BY T. W. WRIGHT, U. S. LAKE SURVEY, DETROIT, MICHIGAN.

IN computing the probable error of the determinations of an observed quantity two forms are in common use, Bessel's and Peters'. If with the ordinary notation we let

m = the number of observations,
r = the probable error of a single observation,
r_0 = the probable error of the final result,
ρ = 0.4769363,
$v_1, v_2 \ldots$ = the residual errors of observation, $[vv]$ the sum of their squares and [v their sum without reg'd to sign, then accord'g to Bessel's form,